# Continuous Piecewise Affine Dynamical Systems do not Exhibit Zeno Behavior

Thuan, Le Quang; Camlibel, M. Kanat

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# Continuous Piecewise Affine Dynamical Systems do not Exhibit Zeno Behavior

Le Quang Thuan and M. Kanat Camlibel

***Abstract*—In the context of continuous piecewise affine dynamical systems, we study the Zeno behavior, i.e., infinite number of mode transitions in finite time interval, in this note. The main result reveals that piecewise affine dynamical systems do not exhibit Zeno behavior. A direct benefit of the main result is that one can apply smooth ordinary differential equations theory in a local manner for the analysis of piecewise affine systems.**

***Index Terms*—Hybrid systems, piecewise affine systems, Zeno behavior.**

## I. Introduction

Analysis, simulation and design of hybrid dynamical systems become considerably complicated when there are infinitely many mode transitions in a finite time interval. Such behavior is called *Zeno* behavior in the literature [14] and [15]. To the best of our knowledge, the earliest work goes back to the eighties when [7] and [22] studied Zeno behavior in the setting of piecewise analytic systems. With the increasing attention to hybrid systems, the study of Zeno behavior received considerable interest in the past few years [1]–[5], [19] and [20].

In this note, we focus on piecewise affine dynamical systems. Piecewise affine dynamical systems are a special kind of finite-dimensional, nonlinear input/state/output systems, with the distinguishing feature that the functions representing the systems differential equations and output equations are piecewise affine functions. Any piecewise affine system can be considered as a collection of finite-dimensional linear input/state/output systems, together with a partition of the product of the state space and input space into polyhedral regions. Each of these regions is associated with one particular linear system from the collection. Depending on the region in which the state and input vector are contained at a certain time, the dynamics is governed by the linear system associated with that region. Thus, the dynamics switches if the state-input vector changes from one polyhedral region to another. Any piecewise affine systems is therefore also a hybrid system.

This note aims at providing conditions guaranteeing the absence of Zeno behavior for a class of piecewise affine dynamical systems. More specifically, we show the absence of Zeno behavior for continuous piecewise affine systems. Similar conditions were already given for various subclasses of piecewise affine systems. The papers [9], [18] and [23] have provided such conditions for linear passive complementarity systems, [12] and [13] for linear complementarity systems with singleton property, [17] for conewise linear systems, [8] for well-posed bimodal piecewise linear systems. Conditions for presence of Zeno behavior have been addressed in [6] and [21] for linear relay systems. Closely related to piecewise affine dynamical systems, differential variational systems were another subclass of hybrid systems for which Zeno behavior has been studied [11] and [16].

The organization of the paper is as follows. In Section II, we introduce continuous piecewise affine dynamical systems and its alternative representations. This will be followed by stating the main result in Section III. Section IV is devoted to the proof of the main result as well as the auxiliary results. Finally, conclusions and future work are addressed in Section V.

## II. Continuous Piecewise Affine Systems

Consider the finite-dimensional time invariant system given by the ordinary differential equation of the form

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous piecewise affine function, i.e., the function $f$ is continuous and there exists a finite family of affine functions $\{f_i \mid 1 \leqslant i \leqslant p\}$ such that

$$f(x) \in \{f_i(x) \mid 1 \leqslant i \leqslant p\}$$

for all $x \in \mathbb{R}^n$.

We say that a continuously differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the system (1) for the initial state $x_0$ if $x(0) = x_0$ and it satisfies the (1) for all $t \in \mathbb{R}$.

The representation (1) describes the system at hand in an implicit way via the component functions $\{f_i \mid 1 \leqslant i \leqslant p\}$. Alternatively, a more explicit representation of (1) can be obtained by invoking the well-known properties of continuous piecewise affine functions. To do so, we first recall that a finite collection $\Xi$ of polyhedra in $\mathbb{R}^n$ is called a polyhedral subdivision of $\mathbb{R}^n$ if

- the union of all polyhedra in $\Xi$ is equal to $\mathbb{R}^n$;
- each polyhedron in $\Xi$ is of dimension $n$;
- the intersection of any two polyhedra in $\Xi$ is either empty or a common proper face of both polyhedra.

For every continuous piecewise affine function $f$, one can find a polyhedral subdivision $\Xi = \{\mathscr{X}_i \mid 1 \leqslant i \leqslant m\}$ of $\mathbb{R}^n$ and a finite family of affine functions $\{f_i \mid 1 \leqslant i \leqslant m\}$ such that $f$ coincides $f_i$ on $\mathscr{X}_i$ (see for instance [10, Proposition 4.2.1]). Suppose that the affine function $f_i$ is given by $f_i(x) = A_i x + b_i$, and $\mathscr{X}_i$ is given by

$$\mathscr{X}_i = \{x \in \mathbb{R}^n \mid C_i x + d_i \geqslant 0\}$$

where $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{m_i \times n}$, $d_i \in \mathbb{R}^{m_i}$. With these definitions, the system (1) can be rewritten in the form

$$\dot{x} = \begin{cases} A_1 x + b_1 & \text{if } C_1 x + d_1 \geqslant 0 \\ A_2 x + b_2 & \text{if } C_2 x + d_2 \geqslant 0 \\ \vdots & \vdots \\ A_m x + b_m & \text{if } C_m x + d_m \geqslant 0. \end{cases} \tag{2}$$

In this case, the continuity of the function $f$ is equivalent to the following implication holds:

$$x \in \mathscr{X}_i \cap \mathscr{X}_j \Rightarrow A_i x + b_i = A_j x + b_j. \tag{3}$$

Since a continuous piecewise affine function must be globally Lipschitz continuous (see for instance [10, Proposition 4.2.2]), it follows from the theory of first-order ordinary differential equations that the system (1)

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L. Q. Thuan is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen 9700 AV, The Netherlands and also with the Department of Mathematics, Quy Nhon University, Binh Dinh, Vietnam (e-mail: t.q.le@rug.nl).

M. K. Camlibel is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Groningen 9700 AV, The Netherlands and also with the Department of Electronics and Communication Engineering, Dogus University, Istanbul 34722, Turkey (e-mail: m.k.camlibel@rug.nl).

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must admit a unique solution for each initial state $x_0$, which is denoted by $x(t;x_0)$.

## III. Main Result

In the hybrid systems literature, the occurrence of an infinite number of mode transitions within a finite time interval is called the Zeno behavior. Next, we define several notions of Zeno behavior for continuous piecewise affine dynamical systems.

*Definition 3.1:* We say that the system (2) has

1) the forward non-Zeno property if for any $x_0 \in \mathbb{R}^n$ and any $t^* \in \mathbb{R}$, there exist $\varepsilon > 0$ and an index $i \in \{1,\ldots,m\}$ such that $x(t;x_0) \in \mathscr{X}_i$ for all $t \in [t^*, t^*+\varepsilon]$.
2) the backward non-Zeno property if for any $x_0 \in \mathbb{R}^n$ and any $t^* \in \mathbb{R}$, there exist $\varepsilon > 0$ and an index $i \in \{1,\ldots,m\}$ such that $x(t;x_0) \in \mathscr{X}_i$ for all $t \in [t^*-\varepsilon, t^*]$.
3) the non-Zeno property if it has both the forward and the backward non-Zeno property.

The following theorem is the main result of the paper. It shows the absence of Zeno behavior in continuous piecewise affine systems.

*Theorem 3.2:* The system (2) has the non-Zeno property.

The proof of this statement will be given in Section IV.

## IV. Proof of Theorem 3.2

Associated to the system (2) we consider the reverse-time system defined as

$$\dot{x}^r = \begin{cases} -A_1x^r - b_1 & \text{if } C_1x^r + d_1 \geqslant 0 \\ -A_2x^r - b_2 & \text{if } C_2x^r + d_2 \geqslant 0 \\ \vdots & \vdots \\ -A_mx^r - b_m & \text{if } C_mx^r + d_m \geqslant 0. \end{cases} \tag{4}$$

Note that this system is also a continuous piecewise affine system. The following proposition relates the backward non-Zeno property of the system (2) to the forward non-Zeno property of the associated reverse-time system (4). Its proof is straightforward and hence is omitted.

*Proposition 4.1:* The system (2) has the backward non-Zeno property if and only if the associated reverse-time system (4) has the forward non-Zeno property.

In the light of this proposition, to prove Theorem 3.2 it suffices to show that every continuous piecewise affine system of the form (2) has the forward non-Zeno property. To do so, we first need to introduce some nomenclature and some auxiliary results.

For an ordered tuple $a = (a_1, a_2, \ldots, a_k)$, we write $a \succeq 0$ if $a = 0$ or the first non-zero component is positive. If $a \succeq 0$ and $a \neq 0$, then we write $a \succ 0$. Sometimes, we also use the symbols "$\preceq$" and "$\prec$" with the obvious meanings. For a finite collection of $n$-dimensional vectors $z = (z^1, z^2, \ldots, z^k)$, we write $z \succeq (\succ) 0$ if for each $j \in \{1,\ldots,n\}$ it holds that $(z_j^1, z_j^2, \ldots, z_j^k) \succeq (\succ) 0$ where the subscript $j$ denotes the $j$th component of the corresponding vector. The same notations are sometimes used for sequences. For a matrix $M$, we denote the $l$th-row of $M$ by $\mathrm{row}_l(M)$.

Next, we define the set

$$\mathscr{Y}_i := \{x \mid (C_ix + d_i, C_iA_ix + C_ib_i, C_iA_i^2x + C_iA_ib_i, \ldots) \succeq 0\}$$

for each $1 \leqslant i \leqslant m$. It follows from Cayley-Hamilton theorem that

$$\mathscr{Y}_i = \{x \mid (C_ix + d_i, C_iA_ix + C_ib_i, \ldots, C_iA_i^nx + C_iA_i^{n-1}b_i) \succeq 0\}$$

for $1 \leqslant i \leqslant m$. It can be verified that $\mathscr{Y}_i$ is convex and $\mathrm{int}(\mathscr{X}_i) \subseteq \mathscr{Y}_i \subseteq \mathscr{X}_i$ for all $1 \leqslant i \leqslant m$.

We will use the sets $\mathscr{Y}_i$ to characterize the forward non-Zeno property. For this purpose, we first state the following lemma that relates the sets $\mathscr{X}_i$, $\mathscr{Y}_i$, and the behavior of solutions.

*Lemma 4.2:* The following statements are equivalent:

1) $x_0 \in \mathscr{Y}_i$.
2) There exists an $\varepsilon > 0$ such that $x(t;x_0) \in \mathscr{X}_i$ for all $t \in [0,\varepsilon]$.

*Proof:* $1 \Rightarrow 2$: Let $\tilde{x}(t;x_0)$ denote the solution of the affine differential equation $\dot{x} = A_ix + b_i$ for the initial state $x_0$, and let denote $\tilde{y}(t;x_0) := C_i\tilde{x}(t;x_0) + d_i$. It can be seen that $\tilde{y}(t;x_0)$ is analytic on $\mathbb{R}$, $\tilde{y}(0;x_0) = C_ix_0 + d_i$, and

$$\tilde{y}^{(k)}(0;x_0) = C_iA_i^kx_0 + C_iA_i^{k-1}b_i$$

for all $k \geqslant 1$. Since $x_0 \in \mathscr{Y}_i$, for each $l \in \{1, 2, \ldots, m_i\}$ only one of the following two cases is possible:

i) The first case is that

$$\mathrm{row}_l(C_ix_0 + d_i, C_iA_ix_0 + C_ib_i, \ldots, C_iA_i^nx_0 + C_iA_i^{n-1}b_i) = 0.$$

Then, by Cayley-Hamilton theorem, we further get

$$\mathrm{row}_l\left(C_iA_i^kx_0 + C_iA_i^{k-1}b_i\right) = 0$$

for all $k \geqslant n+1$, and hence $\mathrm{row}_l\left(\tilde{y}^{(k)}(0;x_0)\right) = 0$ for all $k \geqslant 0$. This implies $\mathrm{row}_l(\tilde{y}(t;x_0)) = 0$ for all $t$ due to the analyticity of $\mathrm{row}_l(\tilde{y}(t;x_0))$, which follows from the analyticity of $\tilde{y}(t;x_0)$.

ii) The second case is that

$$\mathrm{row}_l(C_ix_0 + d_i, C_iA_ix_0 + C_ib_i, \ldots, C_iA_i^nx_0 + C_iA_i^{n-1}b_i) \succ 0. \tag{5}$$

In this case, we first consider the case where the first element is positive, i.e., $\mathrm{row}_l(C_ix_0 + d_i) > 0$. Then, due to continuity there exists $\varepsilon_l > 0$ such that $\mathrm{row}_l(\tilde{y}(t;x_0)) > 0$ for all $t \in [0,\varepsilon_l]$. Now, suppose that $\mathrm{row}_l(\tilde{y}(0;x_0)) = \mathrm{row}_l(C_ix_0 + d_i) = 0$. Then, it follows from (5) that there exists $1 \leqslant k_l \leqslant n$ such that

$$\mathrm{row}_l\left(\tilde{y}^{(j)}(0;x_0)\right) = \mathrm{row}_l\left(C_iA_i^jx_0 + C_iA_i^{j-1}b_i\right) = 0$$

for all $1 \leqslant j \leqslant k_l - 1$ and

$$\mathrm{row}_l\left(\tilde{y}^{(k_l)}(0;x_0)\right) = \mathrm{row}_l\left(C_iA_i^{k_l}x_0 + C_iA_i^{k_l-1}b_i\right) > 0.$$

Since $\mathrm{row}_l\left(\tilde{y}^{(k_l)}(0;x_0)\right) > 0$, there exists $\varepsilon_l > 0$ such that $\mathrm{row}_l(\tilde{y}^{(k_l)}(t;x_0)) > 0$ for all $t \in [0,\varepsilon_l]$ due to continuity $\mathrm{row}_l\left(\tilde{y}^{(k_l)}(t;x_0)\right)$. This means that the function $\mathrm{row}_l\left(\tilde{y}^{(k_l-1)}(t;x_0)\right)$ strictly increases on $[0,\varepsilon_l]$. Hence

$$\mathrm{row}_l\left(\tilde{y}^{(k_l-1)}(t;x_0)\right) > \mathrm{row}_l\left(\tilde{y}^{(k_l-1)}(0;x_0)\right) = 0$$

for all $t \in [0,\varepsilon_l]$. By repeating this argument, we finally obtain $\mathrm{row}_l(\tilde{y}(t;x_0)) > 0$ for all $t \in [0,\varepsilon_l]$.

Finally, we can conclude that there exists $\varepsilon_l > 0$ such that $\mathrm{row}_l(\tilde{y}(t;x_0)) \geqslant 0$ for all $t \in [0,\varepsilon_l]$ for both cases. Define $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_{m_i}\}$. Then, we have $\tilde{y}(t;x_0) = C_i\tilde{x}(t;x_0) + d_i \geqslant 0$ for all $t \in [0,\varepsilon]$ and hence $\tilde{x}(t;x_0) \in \mathscr{X}_i$ for all $t \in [0,\varepsilon]$. This implies that $\tilde{x}(t;x_0)$ is a solution on the interval $[0,\varepsilon]$ of the system (2) for the initial state $x_0$. Due to the uniqueness of solution, we have $x(t;x_0) = \tilde{x}(t;x_0) \in \mathscr{X}_i$ for all $t \in [0,\varepsilon]$.

$2 \Rightarrow 1$: Suppose that $x(t;x_0) \in \mathscr{X}_i$ for all $t \in [0,\varepsilon]$. Then, we have

$$C_ix(t;x_0) + d_i \geqslant 0, \text{ and} \tag{6}$$

$$\dot{x}(t;x_0) = A_ix(t;x_0) + b_i \tag{7}$$

for all $t \in [0, \varepsilon]$. It follows from (7) that $x(t; x_0)$ is analytic on $[0, \varepsilon]$. So, it can be expressed in Taylor series around $t = 0$ as

$$x(t; x_0) = x(0; x_0) + t\dot{x}(0; x_0) + \frac{t^2}{2}x^{(2)}(0; x_0) + \cdots \qquad (8)$$

for all $t \geqslant 0$ sufficiently small. In views of (6) and (8), together with noticing that $x(0; x_0) = x_0$, $x^{(k)}(0; x_0) = A_i^k x_0 + A_i^{k-1} b_i$ for all $k \geqslant 1$, we get

$$0 \leqslant (C_i x_0 + d_i) + t(C_i A_i x_0 + C_i b_i) + \cdots + \frac{t^n}{n!}\left(C_i A_i^n x_0 + C_i A_i^{n-1} b_i\right) + \cdots$$

for all $t \geqslant 0$ sufficiently small. This implies that

$$\left(C_i x_0 + d_i, C_i A_i x_0 + C_i b_i, \ldots, C_i A_i^n x_0 + C_i A_i^{n-1} b_i\right) \succeq 0$$

and hence $x_0 \in \mathscr{Y}_i$. ■

The following lemma characterizes the forward non-Zeno property of a system of the form (2) in terms of the sets $\mathscr{Y}_i, 1 \leqslant i \leqslant m$.

*Lemma 4.3:* The following two statements are equivalent:
1) The system (2) has the forward non-Zeno property.
2) $\bigcup_{i=1}^{m} \mathscr{Y}_i = \mathbb{R}^n$.

*Proof:* Suppose that the first statement holds. Then, for any $x_0 \in \mathbb{R}^n$ there exist an $\varepsilon > 0$ and an index $i \in \{1, \ldots, m\}$ such that $x(t; x_0) \in \mathscr{X}_i$ for all $t \in [0, \varepsilon]$. This implies that $x_0 \in \mathscr{Y}_i$ due to Lemma 4.2. Hence, we get

$$\mathbb{R}^n \subseteq \bigcup_{i=1}^{m} \mathscr{Y}_i.$$

Since the reverse inclusion is evident, the second statement holds. For the converse, let $x_0 \in \mathbb{R}^n$ and $t^* \in \mathbb{R}$ be given. By the second statement, there exists an $1 \leqslant i \leqslant m$ such that $\bar{x} := x(t^*; x_0) \in \mathscr{Y}_i$. By Lemma 4.2 there exists $\varepsilon > 0$ such that $x(t; \bar{x}) \in \mathscr{X}_i$ for all $t \in [0, \varepsilon]$. This follows that

$$x(t; x_0) = x(t - t^*; \bar{x}) \in \mathscr{X}_i$$

for all $t \in [t^*, t^* + \varepsilon]$. Therefore, the system (2) has the forward non-Zeno property. ■

In the light of Lemma 4.3, showing that

$$\bigcup_{i=1}^{m} \mathscr{Y}_i = \mathbb{R}^n$$

is satisfied for any system of the form (2) is enough for proving Theorem 3.2. Before doing so, we need to introduce some more nomenclature.

In general, a given vector in $\mathbb{R}^n$ may be contained in more than one of the sets $\mathscr{X}_i$ and $\mathscr{Y}_i$. Define for a given $\xi$ the sets $I(\xi), J(\xi)$ as

$$I(\xi) := \{i \in \{1, 2, \ldots, m\} \mid \xi \in \mathscr{X}_i\},$$
$$J(\xi) := \{i \in \{1, 2, \ldots, m\} \mid \xi \in \mathscr{Y}_i\}.$$

Since $\mathscr{Y}_i \subseteq \mathscr{X}_i$, one gets $J(\xi) \subseteq I(\xi)$ for each $\xi \in \mathbb{R}^n$.

The following lemma is the last auxiliary result that will be used in the proof of Theorem 3.2.

*Lemma 4.4:* Let $x_0 \in \mathbb{R}^n$ be given. The following statements hold.
1) The set
$$\bigcup_{i \in I(x_0)} \mathscr{X}_i$$
contains a neighborhood of $x_0$.
2) For any $n$-tuple polynomial $p$ in $t$ with $p(0) = x_0$, there exist $i \in I(x_0)$ and $\varepsilon > 0$ such that $p(t) \in \mathscr{X}_i$ for all $t \in [0, \varepsilon]$.

*Proof:* In order to prove the first statement, let $I^c(x_0)$ denote the complement of $I(x_0)$ with respect to the set $\{1, \ldots, m\}$. For $\delta > 0$, denote $B(x_0, \delta)$ the open ball of $\mathbb{R}^n$ centered at $x_0$ with the radius $\delta$, i.e.

$$B(x_0, \delta) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \delta\}.$$

For a subset $X$ of $\mathbb{R}^n$, we denote $X^c$ the complement of $X$ in $\mathbb{R}^n$. Now, for every $j \in I^c(x_0)$, since $x_0 \notin \mathscr{X}_j$, we have $x_0 \in \mathscr{X}_j^c$. Since $\mathscr{X}_j$ is closed, $\mathscr{X}_j^c$ is open. Hence, there exists $r_j > 0$ such that $B(x_0, r_j) \subseteq \mathscr{X}_j^c$. Denote

$$r := \min\{r_j \mid j \in I^c(x_0)\}.$$

Then, $B(x_0, r)$ is a neighborhood of $x_0$, and

$$B(x_0, r) \subseteq \bigcap_{j \in I^c(x_0)} \mathscr{X}_j^c = \mathbb{R}^n \setminus \bigcup_{j \in I^c(x_0)} \mathscr{X}_j = \bigcup_{i \in I(x_0)} \mathscr{X}_i.$$

For the second statement, observe that the claim is trivial if $p$ is a constant function. So, we only need to prove the case when $p$ is not constant. Arguing by contradiction, we assume that the claim does not hold. Then, for each $i \in I(x_0)$, one of the following statements holds:

a) There exists $\varepsilon > 0$ such that $p(t) \notin \mathscr{X}_i$ for all $t \in (0, \varepsilon)$;
b) There exists an infinite sequence of positive scalars $t_k$ all distinct and converging to 0 as $k \to \infty$ such that $p(t_{2k-1}) \in \mathscr{X}_i$ and $p(t_{2k}) \notin \mathscr{X}_i$ for all $k \geqslant 1$.

Note that the latter statement must hold for at least one index $i \in I(x_0)$. Otherwise, there would exist a positive number $\bar{\varepsilon}$ such that

$$p(t) \notin \bigcup_{i \in I(x_0)} \mathscr{X}_i$$

for all $t \in (0, \bar{\varepsilon})$. However, the set on the right hand side contains a neighborhood $U$ of $x_0$ due to the first statement of Lemma 4.4. This leads to a contradiction since $p(t)$ belongs to $U$ for all sufficiently small $t > 0$ due to continuity.

Since (b) holds for some $i \in I(x_0)$, for every $k \geqslant 1$ there exists an index $l_k \in \{1, \ldots, m_i\}$ such that

$$\mathrm{row}_{l_k}(C_i p(t_{2k}) + d_i) < 0.$$

Denote $\Lambda = \{l_k \mid k \geqslant 1\}$. Then, note that $\Lambda$ is a finite set, there exists an index $l^* \in \Lambda$ such that

$$\mathrm{row}_{l^*}(C_i p(t_{2k}) + d_i) < 0$$

for infinitely many $k$'s. Without loss of generality, we may assume that

$$\mathrm{row}_{l^*}(C_i p(t_{2k}) + d_i) < 0 \qquad (9)$$

for all $k \geqslant 1$. Then, for every $k \geqslant 1$, due to (9) and the fact that $\mathrm{row}_{l^*}(C_i p(t_{2k-1}) + d_i) \geqslant 0$, there exists $\mu_k \in [t_{2k-1}, t_{2k})$ such that

$$\mathrm{row}_{l^*}(C_i p(\mu_k) + d_i) = 0.$$

Since the $\mu_k$'s are all distinct and $\mathrm{row}_{l^*}(C_i p(t) + d_i)$ is a non-zero polynomial in $t$ with real coefficients with finitely many roots, we obtain $\mathrm{row}_{l^*}(C_i p(t) + d_i) = 0$ for all $t$. This is contradiction with (9) and hence the second statement holds. ■

With all these preparations, we are now in a position to prove Theorem 3.2. In views of Proposition 4.1 and Lemma 4.3, it is enough to show that

$$\bigcup_{i=1}^{m} \mathscr{Y}_i = \mathbb{R}^n$$

holds for any system of the form (2). For any choice $x_0 \in \mathbb{R}^n$, we denote $I_0 = I(x_0)$ for brevity. It follows from (3) that $A_i x_0 + b_i = A_j x_0 + b_j$ for all $i, j \in I_0$. We take $i \in I_0$ and define $x_1 := A_i x_0 + b_i$. Lemma 4.4 ensures that the set

$$I_1 := \{i \in I_0 \mid x_0 + t x_1 \in \mathscr{X}_i \text{ for all sufficiently small } t > 0\}$$

is non-empty. Since $I_1 \subseteq I_0$ and $A_i(x_0 + t x_1) + b_i = A_j(x_0 + t x_1) + b_j$ for all $i, j \in I_1$ and for all sufficiently small $t > 0$, one has $A_i x_1 = A_j x_1$ for all $i, j \in I_1$. Let us define $x_2 := A_i x_1$ for some $i \in I_1$. The set

$$I_2 := \{i \in I_0 \mid x_0 + t x_1 + t^2 x_2 \in \mathscr{X}_i \text{ for all sufficiently small } t > 0\}$$

is non-empty due to Lemma 4.4. Now, we claim that $I_2 \subseteq I_1$. Indeed, for any $i \in I_2$ we have $C_i x_0 + d_i \geqslant 0$ because $I_2 \subseteq I_0$. Moreover, if $\mathrm{row}_l(C_i x_0 + d_i) = 0$ for some index $1 \le l \leqslant m_i$ then we must have $\mathrm{row}_l(C_i x_1) \geqslant 0$. Hence, it must hold that $C_i(x_0 + t x_1) + d_i \geqslant 0$ for all sufficiently small $t > 0$. Thus, it follows that $i \in I_1$ and then $A_i x_2 = A_j x_2$ for all $i, j \in I_2$. Next, we define $x_3 := A_i x_2$ for some $i \in I_2$ and

$$I_3 := \{i \in I_0 \mid x_0 + t x_1 + t^2 x_2 + t^3 x_3 \in \mathscr{X}_i \text{ for all sufficiently small } t > 0\}.$$

Due to Lemma 4.4, the set $I_3$ is non-empty. By similar arguments, we can show that $I_3 \subseteq I_2$ and $A_i x_3 = A_j x_3$ for all $i, j \in I_3$. Continuing this process, we can construct a sequence of index sets $I_1, I_2, \ldots, I_n$ and a sequence of elements $x_1, x_2, \ldots, x_n$ such that the inclusions

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots \supseteq I_n \neq \emptyset$$

hold and $x_0 + t x_1 + \cdots + t^n x_n \in \mathscr{X}_i$ for all $i \in I_n$ and for all $t \in [0, \varepsilon)$ for some $\varepsilon > 0$. We now claim that $x_0 \in \mathscr{Y}_i$ for any $i \in I_n$. Indeed, for any $i \in I_n$, the element

$$x_0 + t x_1 + \cdots + t^n x_n = x_0 + t(A_i x_0 + b_i) + \cdots + t^n \left(A_i^n x_0 + A_i^{n-1} b_i\right)$$

is in $\mathscr{X}_i$ for all $t \in [0, \varepsilon)$ by construction. Thus

$$C_i x_0 + d_i + t(C_i A_i x_0 + C_i b_i) + \cdots + t^n \left(C_i A_i^n x_0 + C_i A_i^{n-1} b_i\right) \geqslant 0$$

for all $t \in [0, \varepsilon)$. This implies that

$$\left(C_i x_0 + d_i, C_i A_i x_0 + C_i b_i, \ldots, C_i A_i^n x_0 + C_i A_i^{n-1} b_i\right) \succeq 0$$

and hence $x_0 \in \mathscr{Y}_i$. ■

## V. Conclusions

In this note, we proved that continuous piecewise affine dynamical systems do not exhibit Zeno behavior. Absence of Zeno behavior considerably simplifies the analysis of piecewise affine systems. This opens new possibilities in studying fundamental system-theoretic problems like controllability and observability for these systems. Also the ideas employed in this note are akin to be extended for possibly discontinuous but well-posed (in the sense of existence and uniqueness of solutions) piecewise affine dynamical systems.

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# Cardinality Constrained Linear-Quadratic Optimal Control

Jianjun Gao and Duan Li

***Abstract*—As control implementation often incurs not only a variable cost associated with the magnitude or energy of the control, but also a setup cost, we consider a discrete-time linear-quadratic (LQ) optimal control problem with a limited number of control implementations, termed in this technical note the cardinality constrained linear-quadratic optimal control (CCLQ). We first derive a semi-analytical feedback policy for CCLQ problems using dynamic programming (DP). Due to the exponential growth of the complexity in calculating the action regions, however, DP procedure is only efficient for CCLQ problems with a scalar state space. Recognizing this fact, we develop then two lower-bounding schemes and integrate them into a branch-and-bound (BnB) solution framework to offer an efficient algorithm in solving general CCLQ problems. Adopting the devised solution algorithm for CCLQ problems, we can solve efficiently the linear-quadratic optimal control problem with setup costs.**

***Index Terms*—Branch-and-bound (BnB), cardinality constraint, dynamic programming, linear-quadratic (LQ) control, quadratic programming, semidefinite programming (SDP), setup cost.**

## I. Introduction

The success of the linear-quadratic (LQ) optimal control is one of the most remarkable achievements in control theory, largely due to its mathematical elegance in tractability and a wide range of its applications. The past three decades have witnessed many extensions of the conventional LQ control, see, e.g., [1]–[5]. Recent years have also observed promising applications of semidefinite programming (SDP) in solving constrained LQ optimal control problems [6], [7].

Implementation of a control action often incurs two types of costs, fixed (setup) costs and variable costs associated with the magnitude or energy of the control. This is especially true for control problems in economics and management science [8], [9]. While the variable cost has already been investigated in the LQ control literature, the setup cost has not yet been addressed, even when it deems necessary in problems such as the LQ control problem for a closed economy with bond-financed budget deficit [8]. To solve the LQ optimal control problem with setup cost, we investigate in this technical note the cardinality constrained LQ optimal control problem(CCLQ), i.e., an LQ optimal control problem with limited number of control implementations. We derive first explicitly the feedback policy and the action region for CCLQ problems using dynamic programming (DP). Recognizing that DP procedure is only efficient for CCLQ problems with a scalar state space due to the NP hardness of CCLQ, we develop then two lower-bounding schemes and integrate them into a branch-and-bound (BnB) solution framework to offer an efficient algorithm in solving general CCLQ problems. Adopting the devised solution algorithm for CCLQ, we finally solve discrete-time linear-quadratic optimal control problem with setup costs.

## II. Problem Formulation

We are interested in the following discrete-time LQ optimal control problem with setup costs

$$(\mathcal{P}): \quad \min_{u_t} \quad \sum_{t=0}^{T-1} \left\{ w\sigma(u_t) + \left[ x'_{t+1}Q_{t+1}x_{t+1} + u'_tR_tu_t \right] \right\},$$
$$\text{Subject to}: x_{t+1} = A_tx_t + B_tu_t, \quad t = 0, 1, \ldots, T-1 \tag{1}$$

where the model setting and assumptions are the same as the conventional LQ optimal control, except that $w > 0$ is the setup cost of implementing a control action and $\sigma(\cdot)$ is the indicator function, i.e., $\sigma(u_t) = 0$ if $u_t$ is a zero vector and $\sigma(u_t) = 1$ otherwise. Setup cost $w$, in many situations, prevents control actions from being implemented at every time period. To solve problem $(\mathcal{P})$, we consider the following cardinality constrained LQ optimal problem (CCLQ)

$$\left(\tilde{\mathcal{P}}(s)\right): \quad \min \quad \sum_{t=0}^{T-1} \left[ x'_{t+1}Q_{t+1}x_{t+1} + u'_tR_tu_t \right], \tag{2}$$
$$\text{Subject to}: \quad \{x_t, u_t\} \text{ satisfies } (1),$$
$$\sum_{t=0}^{T-1} \sigma(u_t) \le s \tag{3}$$

where parameter $s \le T$ is a given positive integer. Throughout the technical note, we use $v(\cdot)$ to denote the optimal value of a given problem $(\cdot)$, $Q \succeq 0$ $(Q \succ 0)$ a positive semidefinite (positive definite) matrix, $\mathbb{S}^n_+$ the set of positive semidefinite matrices, and $\mathrm{diag}\{S_t\}|_{t=0}^{T-1}$ the block diagonal matrix with matrix element $S_t$ for $t = 0, 1, \ldots, T-1$. The following lemma is obvious.

*Lemma 2.1:* For any $s_1$ and $s_2$ that satisfy $1 \le s_1 < s_2 \le T$, $v(\tilde{\mathcal{P}}(s_1)) \ge v(\tilde{\mathcal{P}}(s_2))$.

The optimal solution for problem $(\mathcal{P})$ can be found by identifying the optimal cardinality $s^*$ such that $s^* \triangleq \arg\min_{0 \le s \le T}\{ws + v(\tilde{\mathcal{P}}(s))\}$. It is evident that the optimal control of $(\tilde{\mathcal{P}}(s^*))$ also solves $(\mathcal{P})$. Thus, an efficient solution scheme of problem $(\tilde{\mathcal{P}}(s))$ plays a key role in solving problem $(\mathcal{P})$. Some preliminary results for CCLQ were reported in [10] and [11].

Note that CCLQ problem can be also examined from the view point of optimal control of the linear switched systems [12]–[14] and hybrid systems [15]–[17], at least formally. In most of the literature on hybrid systems, for example piece-wise affine system (PWA) [17], switches among different operating modes are governed by inherent system characteristics, especially triggered when crossing the boundary of a given partition in the state/control space, resulting in possible mixed

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The authors are with the Department of Systems Engineering and Engineering Management, Chinese University of Hong Kong, China (e-mail: dli@se.cuhk.edu.hk).

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